Mathematical and Computer Simulations of Stochastic Processes of Electron Multiplication.

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Abstract

This paper is devoted to a theoretical investigation of stochastic processes of an electron multiplication. The developed method is based on Monte Carlo simulations and theorems about series and parallel amplification stages proposed here. Splitting a stochastic process into a number of different stages, enables a contribution of each stage to the entire process to be easily investigated. In such approach, Monte Carlo simulations are used only once for one simple stage. The use of the theorems provides a high calculation accuracy with minimal cost of computations. The method is especially efficient for optimization problems which require computer simulations. In this paper the method is used to investigate the effect of variations in channel diameters on noise characteristics of microchannel electron multipliers.

Keywords: Stochastic process, electron multiplication, Monte Carlo simulations, microchannel electron multiplier, noise factor.

1 Introduction

Amplification of the elementary particles in electronic devices, particularly in microchannel electron multipliers, is a complicated stochastic process,

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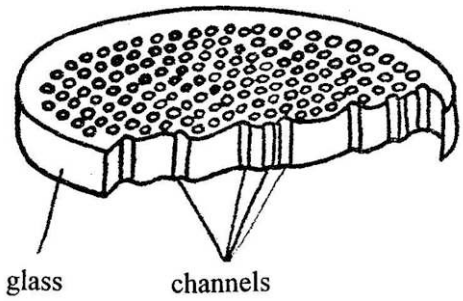

Figure 1: Microchannel plate

which is usually simulated by the use of Monte Carlo (MC) methods ([1], [2], [6], [7], [9], [11]). However, for optimization problems and some investigations, the direct simulation of the entire amplification process by the MC methods requires considerable computer time. Moreover, it is quite problematic to perform some investigations and optimizations using only MC simulations.

The essence of the approach proposed here consists of separating the amplification process into series and parallel stages. The advantage of the method is that MC simulations are used once for one simple stage of the complicated stochastic process. Any further investigations and optimizations do not require any additional use of the MC methods. Moreover, splitting a stochastic process into a number of different stages, allows a contribution of each stage to the entire process to be easily investigated.

In this paper the model is used to investigate the effect on noise characteristics of the microchannel plate caused by the variations of the channel diameters (see Fig. 1 which is taken from [14]). Microchannel plates have found wide applications in different areas of science, engineering, medicine etc. However, the loss of information caused by the statistical fluctuations in the gain of the channels, and by loss of primary electrons when they strike the closed area of a channel plate increases a noise factor [7], [10], [17].

Section 2 includes a proof of the theorem of series amplification stages and a proof of the theorem of n parallel amplification paths. Expressions for the mean and variance of the amplitude distribution at the output of the entire system are obtained.

Section 3 describes a computational model and an algorithm of the electron multiplication inside the channel multiplier. The Monte Carlo simulation procedure is briefly described.

It is also shown that the amplitude distribution at the output of the channel is determined by the effective length of the channel, where the am-

plitude distribution is stabilized. The MC simulations are carried for one electron emitted at the beginning of the channel, along the effective length. The gain and variance as functions of the distance from the channel entrance are calculated. The use of the theorems allows the output characteristics for the entire system to be obtained.

Section 4 shows the efficiency and accuracy of the proposed method. It compares some theoretical and experimental data.

In Section 5, expressions for the mean and variance of the amplitude distribution at the output of an array of channels, and an expression for the noise factor are obtained.

Section 6 is devoted to evaluation of the effect of variations in channel diameters in a channel plate on the noise factor.

2 Theorems about Series and Parallel Amplification Stages

2.1 Theorem about series amplification stages

Let $p_k(\nu)$ be the probability distribution of the number of particles at the output of the k-th stage, produced by one particle from the $(k-1)$-th stage. Then the generating function of the probability distribution $p_k(\nu)$ is:

$$g_k(u) = \sum_{\nu=0}^{\infty} u^{\nu} p_k(\nu) \qquad where \qquad |u| \leq 1.$$

Using an approach similar to [16] the generating function for the probability distribution of the number of particles after the last (N-th) stage can be constructed as:

$$G_N(u) = G_{N-1}[g_N(u)] \qquad or \qquad G_N(u) = q_0(q_1(q_2(...(g_N(u))...))) \quad (1)$$

To find the mean M, and variance D of the amplitude distribution $P_N(\nu)$ after the N-th stage, we convert the expressions (1) to the logarithmic generating functions, introducing new variables:

$$v = \ln u, \qquad h_k(v) = \ln \sum_{\nu=0}^{\infty} e^{v\nu} p_k(\nu) \qquad H_N(v) = \ln \sum_{\nu=0}^{\infty} e^{v\nu} P_N(\nu)$$

Then the expressions (1) can be written as:

$$H_N(v) = H_{N-1}[h_N(v)] \qquad or \qquad H_N(v) = h_0(h_1(h_2(...(h_N(v))...)))$$

where $H_N(v)$ is the logarithmic generating function of the distribution $P_N(\nu)$ of the number of particles after the N-th stage; $h_k(v)$ is the logarithmic generating function of the distribution $p_k(\nu)$ of the number of particles at the output of the k-th stage, produced by one particle from the $(k-1)$-th stage.

Differentiating $H_N(v)$ with respect to v once and using the properties of the logarithmic generating functions, with $v = 0$ we obtain:

$$M = m_0 m_1 ... m_k ... m_N = \prod_{k=0}^{N} m_k \tag{2}$$

where m_k is the mean value of the distribution of the number of particles at the output of the k-th stage for one particle at its input, and M is the mean of the amplitude distribution after the last N-th stage.

Differentiating $H_N(v)$ with respect to v twice, with $v = 0$ we obtain the variance D after the N-th stage of this multistep sequential process.

$$\begin{aligned} D &= d_0(m_1 m_2 ... m_N)^2 + d_1 m_0 (m_2 m_3 ... m_N)^2 + d_2 m_0 m_1 (m_3 m_4 ... m_N)^2 + ... \\ &... + d_k m_0 m_1 ... m_{k-1} (m_{k+1} m_{k+2} ... m_N)^2 + ... + d_N m_0 m_1 ... m_{N-1} \end{aligned}$$

or

$$D = \sum_{k=0}^{N} d_k \prod_{i=0}^{k-1} m_i \prod_{j=k+1}^{N} m_j^2 \tag{3}$$

The expressions (2) and (3) constitute the theorem of series amplification stages. The theorem is a generalization of the Burgess theorem [17] for a multistep sequential process. The relations (2) and (3) are transformed to the Burgess theorem for N=1.

2.2 Theorem of parallel amplification paths

Let the primary particle be multiplied along one of n possible parallel paths, and p_k be the probability of choosing the k-th path. If each path gives an average of g_k particles at the output with a variance of d_k, then the mean G and the variance D of this multiplication process can be obtained.

Let $\varphi(\nu)$ be the probability distribution of the number of particles ν at the output of the k-th path produced by one particle at its input. Then the

probability distribution $\Phi(\nu)$ of the number of particles at the output of the entire system of n parallel paths will be:

$$\Phi(\nu) = \sum_{k=1}^{n} p_k \varphi(\nu)$$

Then the mean G of such a multiplication process is equal to:

$$G = \sum_{\nu=0}^{\infty} \Phi(\nu)\nu = \sum_{k=1}^{n} p_k \sum_{\nu=0}^{\infty} \varphi(\nu)\nu = \sum_{k=1}^{n} p_k g_k \tag{4}$$

The variance D of the distribution at the output of the system can be written as:

$$D = \sum_{\nu=0}^{\infty} \Phi(\nu)\nu^2 - [\sum_{\nu=0}^{\infty} \Phi(\nu)\nu]^2 \tag{5}$$

The first sum in (5) can be transformed to:

$$\sum_{\nu=0}^{\infty} \Phi(\nu)\nu^2 = \sum_{k=1}^{n} p_k d_k + \sum_{k=1}^{n} p_k g_k^2$$

Taking into account that $[\sum_{\nu=0}^{\infty} \Phi(\nu)\nu]^2 = G^2$ finally the expression

$$D = \sum_{k=1}^{n} p_k d_k + \sum_{k=1}^{n} p_k g_k^2 - G^2 \tag{6}$$

is obtained, where G is determined by (4).

Equations (4) and (6) can be used for discrete and for continuous systems. For example, an individual channel in an assembly, where there is a spread in the channel diameters, can be taken as one amplification path. Also, variations in the collision coordinates of the electrons of the primary beam can be considered as a choice of the amplification path. The latter case is the example of the continuous amplification system, and the equations (4) and (6) should be written as:

$$G = \int_s \psi(s) g(s) ds \tag{7}$$

and

$$D = \int_s \psi(s) d(s) ds + \int_s \psi(s) g^2(s) ds - G^2 \tag{8}$$

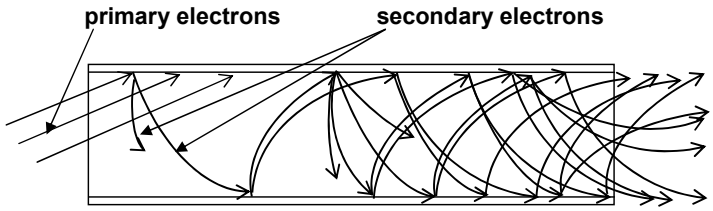

Figure 2: Electron multiplication in the channel

where s is the surface area stroked by particles; ψ is the probability density for the particle to strike the elementary surface ds; $g(s)$ is the average number of particles with variance $d(s)$ at the output of the path. Equations (4), (6), (7) and (8) constitute the theorem of parallel amplification paths.

3 Computer Model

3.1 Monte Carlo simulations

The following physical picture was considered in the modelling. The electrons of a parallel monochromatic beam are incident on the input plane of a microchannel multiplier. Electrons entering the channel have different incidence coordinates and hit the walls at different angles, producing secondary electrons with different emission energy and directions. The secondary electrons are multiplied until they leave the channel (Fig.2).

A secondary emission function [7] is used in the computer model to represent the variation of the secondary electron emission coefficient (SEEC)

$$\sigma = \sigma_m[\frac{V}{V_m}(\cos\theta)^{1/2} exp\{\alpha(1-\cos\theta) + \beta[1 - \frac{V}{V_m}(\cos\theta)^{1/2}]\}$$

where σ_m is the maximum SEEC, V_m is the collision energy in eV which is equivalent to σ_m. Both V_m and σ_m are functions of the collision angle θ ($\theta = 0$ at normal incidence); α and β are constants of the channel multiplier surface , and are chosen to fit experimental secondary emission curves at normal incidence. (The values $\sigma_m = 3.15$, $V_m = 300eV$ for $\theta = 0$, $\alpha = 0.62$ and $\beta = 0.6$ were used throughout the simulations).

The actual number of secondaries generated by the particular collision is a random sample taken from the Poisson distribution:

$$P(\nu) = \frac{\sigma^\nu e^{-\sigma}}{\nu!}$$

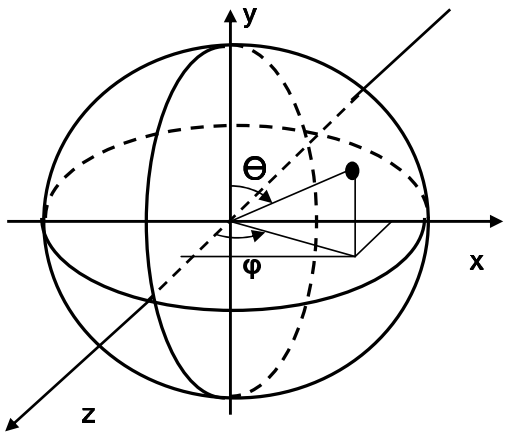

Figure 3: Emission angles of the secondary electrons

where ν is the number of secondary electrons produced, σ is SEEC, calculated according to the formula above.

The energy distribution is described by a Yakobson formula [15]:

$$p(\varepsilon) = 2.1\bar{\varepsilon}^{-3/2}\sqrt{\varepsilon}exp(-1.5\varepsilon/\bar{\varepsilon})$$

where $\bar{\varepsilon}$ is the mean energy (the value $\bar{\varepsilon} = 5eV$ were used throughout the simulations).

Each secondary electron is assigned two emission angles chosen from Lambert's law (Fig.3):

$$p_1(\theta) = \sin 2\theta \qquad p_2(\varphi) = 1/2\pi$$

The trajectory of each electron is calculated in three dimensions from the ballistic equations, and so the position, energy, and angle of the subsequent collisions are determined. The result of each collision is calculated as before and the process is repeated for each secondary electron generated.

A contact conducting layer is deposited at the entrance of the channel multiplier. Since there is no potential drop along the conducting layer, the conditions for the movement of secondary electrons in this region are drastically different from motion in a uniform field. Therefore, the electron amplification process has characteristics different from those usually considered in a uniform field.

The trajectories of the electrons in the nonuniform field are calculated by solving the system of differential equations of motion in fields having axial symmetry by the Runge-Kutta method [3], [5], [8]. The trajectories of the electron motion inside the channel are calculated from the equations of motion in the uniform field.

The total number of electrons which have left the output of the channel is accumulated continually. When all the electrons are calculated to have emerged from the channel, the yield of the individual pulse is known. The process is repeated many times to produce a series of output pulses.

3.2 Effective length of the channel

It can be shown that the form of the pulse distribution at the output of the channel is determined by its initial section (the effective length l_{eff}).

Formulas (2) and (3) enable one to evaluate the number of stages n, after which the relative variance has an error δ compared with the relative variance of the amplitude distribution at the output of the entire channel.

Let stages from $(l+1)$ to the end of the channel have the same mean m and variance d each. If M_l is the mean and D_l is the variance of the amplitude distribution after l first stages, then according to (2) and (3) the expressions for M and D at the output of the entire system can be obtained:

$$M = M_l m^k$$

$$D = D_l m^{2k} + dM_l m^{k-1} \frac{m^k - 1}{m - 1}$$

where k is the number of the last amplification stages.

Suppose $m^k \gg 1$ then the formula for the relative variance v_r will be:

$$v_r = \frac{D}{M^2} = \frac{D_l}{M_l^2} + \frac{d}{(m-1)M_l m}$$

The relative variance v_{rn} after n similar stages will be:

$$v_{rn} = \frac{D_l}{M_l^2} + \frac{d(m^n - 1)}{(m-1)M_l m^{n+1}}$$

The absolute value of the error δ of the relative variance after n stages compared with the relative variance at the output of the system is:

$$\delta = \frac{|v_{rn} - v_r|}{v_r} = \frac{dM_l}{D_l(m-1)m + dM_l} . \frac{1}{m^n}$$

Since $D_l > M_l$ we obtain:

$$\delta < \frac{d}{(m-1)m + d} . \frac{1}{m^n}$$

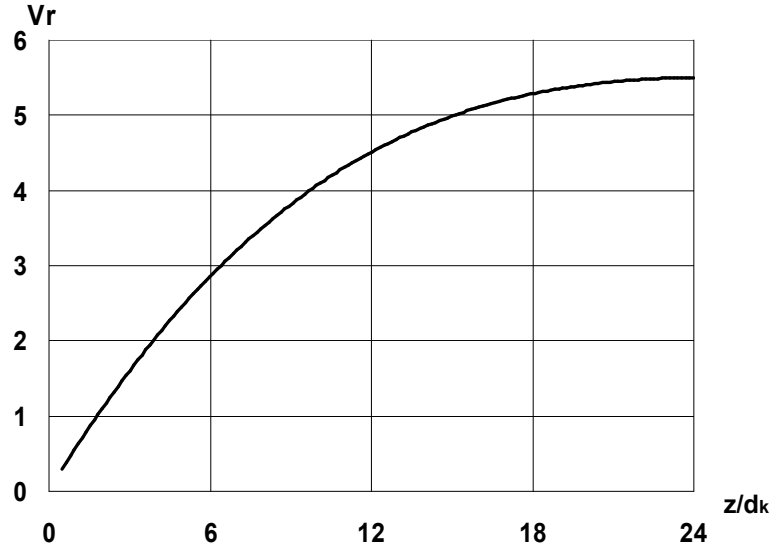

Figure 4: The relative variance as a function of the channel length

Assuming that each separate stage of amplification in the channel is described by a negative exponential distribution (Furry's statistics) [10] with a mean gain of m and variance $d = m(1+m)$ we obtain the number of stages n after which the relative variance has an error δ compared with the relative variance at the output of the entire channel:

$$n < \ln(\frac{1+m}{2m\delta}) / \ln m$$

At the same time the effective length l_{eff} can be evaluated as $l_{eff} = \lambda n$ where λ is the average free path of electrons in the channel.

For $\delta = 0.01$, for typical values of the multiplier parameters, l_{eff} corresponds to half the channel length. The numerical experiment, using the MC methods, completely confirms this result (Fig.4).

Fig.4 shows the relative variance v_r as a function of the length of the channel. It has been calculated for a single electron emitted at the beginning of the channel (z is the length of the channel, and d_k is its diameter.)

The effective length can be defined as a part of the channel where the amplitude distribution is stabilized, and the shape of the distribution is close to a negative exponential function. Figures 5 and 6 show the amplitude distributions calculated by MC methods, for the different lengths of the channel.

For the small length ($z/d_k = 1$) the distribution is close to the Poisson distribution. As the length of the channel is increasing ($z/d_k = 22$, which is relevant to the half of the channel) the distribution changes to the negative

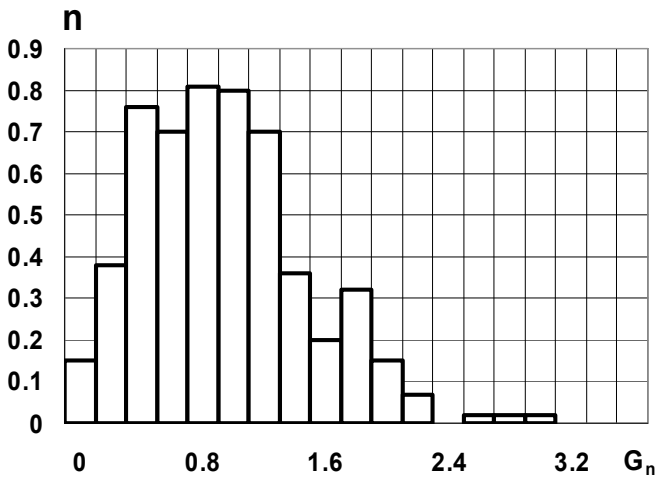

Figure 5: The amplitude distribution for $z/d_k = 1$

exponential function. (These results are obtained for the channel diameter $d_k = 10\mu m$ and a voltage on the channel $V = 800\ V$.)

3.3 Computational algorithm

The multiplication process of a single electron with an initial departure coordinate of $z = 0$ (z is the coordinate directed along the channel axis and measured from its beginning) is simulated by MC methods in a homogeneous field along half the channel length. Functions $g(z)$, the mean, and $d(z)$, the variance, are calculated for $0 \leq z \leq L/2$, where L is the coordinate of the end of the channel. For n electrons leaving the first half of the channel, the incidence coordinates ($z > L/2$) and the values of the SEEC (σ) are determined.

The amplification in the second half of the channel is considered to consist of n parallel paths. Each path has two sequential stages: first collision and multiplication of a single electron until it leaves the channel. Using the theorems of series and parallel amplification stages, the functions $g(z)$ and $d(z)$ along the entire channel length ($0 \leq z \leq L$) are calculated.

The functions $g(z)$ and $d(z)$ for $0 \leq z \leq L$ and the theorems of series and parallel amplification stages allow us to conduct further investigations and optimizations without any additional MC simulations and provide highly accurate results.

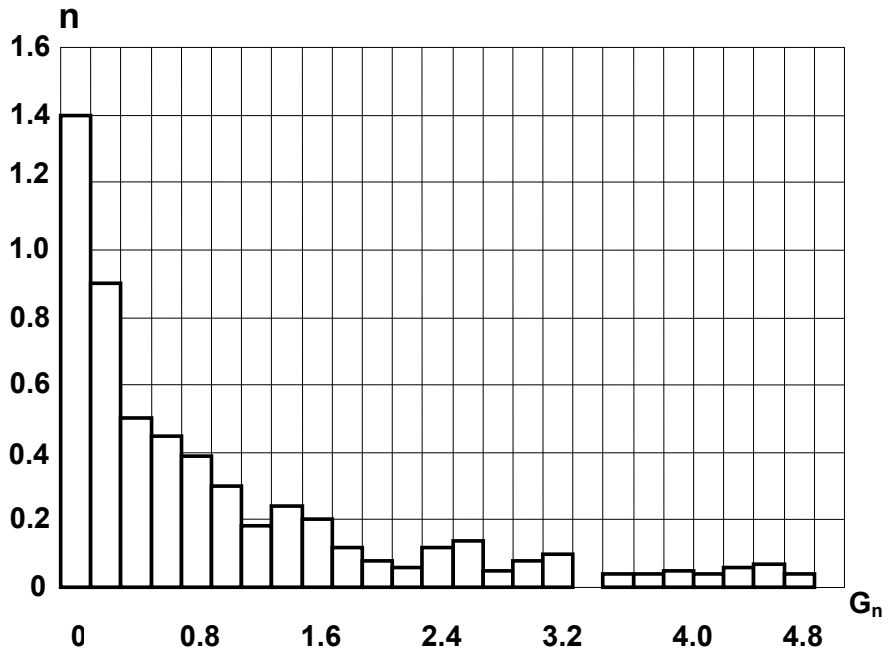

Figure 6: The amplitude distribution for $z/d_k = 22$

4 Evaluation of the Efficiency of the Method

The mean gain on the channel length z can be defined [17] as :

$$G(z) = e^{\alpha z} \tag{9}$$

where α is the electron gain on the length unit.

Let $x = z/d$, where d is the channel diameter. Then the time needed to calculate one electron pulse is proportional to the number of the emitted electrons. An average number of collisions on the length dx is proportional to $dG/dx = \alpha e^{\alpha x}$. Then the time needed to calculate the electron pulse can be declared as

$$\tau = \tau_0 \int_0^x \alpha e^{\alpha t} dt = \tau_0 (e^{\alpha x} - 1) \tag{10}$$

Values of τ_0 and α can be defined from computational experiments. Fig.7 shows the average time needed for MC simulations of one electron pulse as a function of the channel length. (This data is relevant for the computer Pentium 4). From the graph, $\tau_0 = 0.44\ msec$ and $\alpha = 0.12$.

Simple calculations show that MC simulations of one amplitude distribution at the output of the channel with a spread in the collision coordinates of the input beam of electrons (which is the practical case) will take more

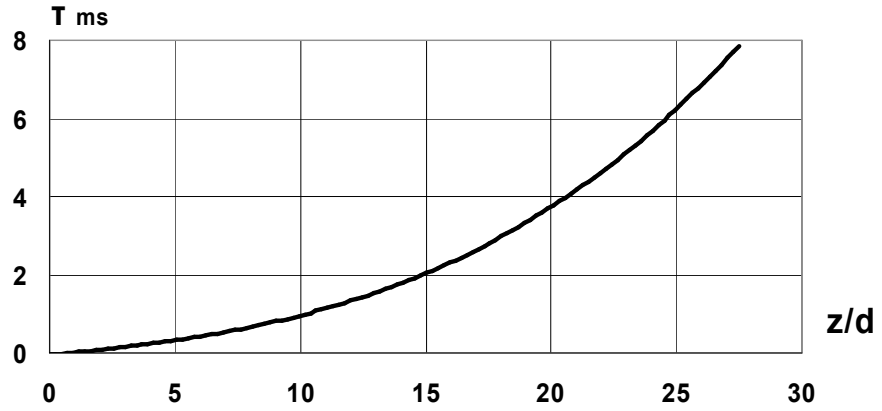

Figure 7: Average time taken for simulation of one electron pulse

than 5 hours. The use of the theorems proposed here and the algorithm described above reduces the time requirements to 5 - 10 seconds.

Calculations of the dependence of the gain and the noise factor (Fig. 8 and 9) on the energy of the input electron beam would take about 3 days and nights of the constant work of the computer for one characteristic. The use of the proposed theorems reduces the cost of calculations to 30 - 60 seconds.

It would require about 20 days and nights to find the optimal combination of the energy and the angle of the input electron beam which provides the minimal noise factor [4], [13] but about 1 - 2 minutes if the proposed theorems are used.

These evaluations are done for the uniform electrostatic field at the channel entrance. For the nonuniform electrostatic field the cost of calculations will be increased significantly for the direct MC simulations. Moreover, if some parameters of the input beam or of the channel were changed, the MC simulation should be conducted again.

Alternatively, the method proposed here does not require additional use of MC methods. The MC simulations in this case, should be conducted only once on the effective channel length for one electron emitted at the beginning.

The figures 8 and 9 demonstrate the accuracy of the method. They show the dependence of the noise factor and the average gain on the energy of the input electron beam. The theoretical results (solid curves) are compared with the experimental data (dashed curves) [4]. (Here, the diameter of the channel is $d = 10\mu m$, the voltage $V = 800\ V$, and the angle of the input beam $\theta = 85^0$.)

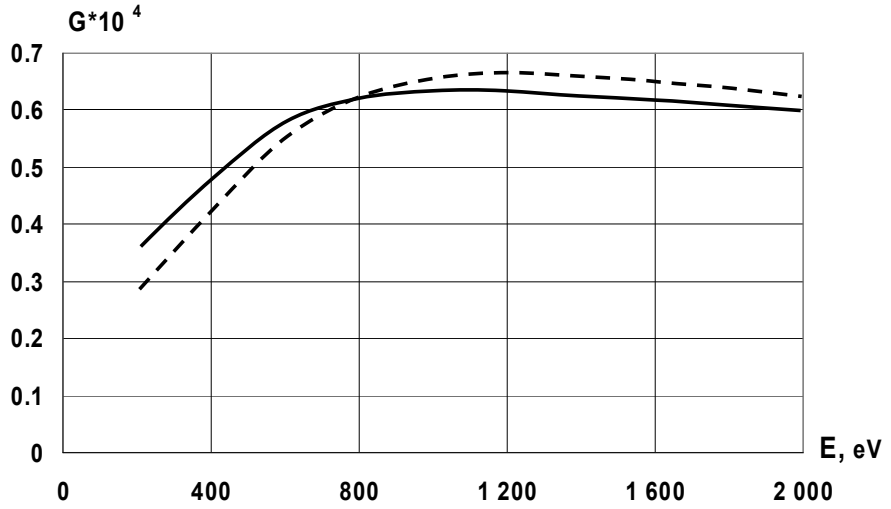

Figure 8: Dependence of the average gain on the energy of the input electron beam (solid curve represents calculations and dashed curve represents experimental results).

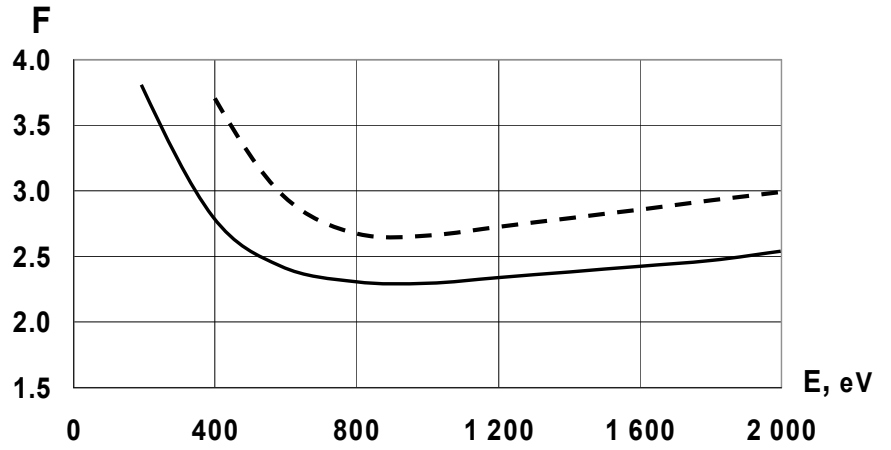

Figure 9: Dependence of the noise factor on the energy of the input electron beam (solid curve represents calculations and dashed curve represents experimental results).

5 The Noise Factor of the Channel Multiplier

5.1 The noise factor of a single channel

Whenever the channel multiplier is used in a device, some of the information potentially available is lost [7]. The noise factor F, which is a measure of the loss of available information can be written as

$$F = \frac{(S/N)_{in}^2}{(S/N)_{out}^2} \tag{11}$$

where $(S/N)_{in}$ and $(S/N)_{out}$ are ratios of the input signal to the noise and the output signal to the noise respectively.

Assuming that the number of electrons entering the multiplier fluctuates according to Poisson's law [10] with the mean n, we obtain

$$(S/N)_{in}^2 = (n/\sqrt{n})^2 = n \tag{12}$$

Using the definition of the noise factor (11) and the theorems about series amplification stages and parallel amplification paths expressions for calculating the noise factor can be obtained. The expressions depend on how the entire process is split into a sequence of amplification steps.

The entire amplification process in the channel multiplier can be represented in the form of a sequence of the next several stages.

1. The first observation of electrons, incident at the input of the multiplier (described by the Bernoulli distribution [10]), can be defined as a first stage. If γ is the fraction of the front surface of the multiplier exposed to electrons, then the average number of particles entering the channel and the variance can be given by

$$m_1 = \gamma \qquad\qquad d_1 = \gamma(1-\gamma) \tag{13}$$

2. The collision of the primary electrons with the wall of the channel is defined as the second stage of the amplification. The distribution $P(\nu)$ (with the mean m_2 and the variance d_2) of the number of electrons knocked out by one primary electron depends on the properties of the emitter. For a uniform emitter the number of electrons fluctuates according to Poisson's law with

$$m_2 = d_2 = \sigma_1$$

where σ_1 is the coefficient of the secondary emission of the emitting surface.

3. Further amplification of the electrons in the channel is regarded as the third stage with the mean gain $m_3 = g(L)$ and the variance $d_3 = d(L)$.

Taking into account the contribution of each stage to the overall process of amplification and with the help of (2) and (3) we obtain:

$$(\frac{S}{N})^2_{out} = \frac{[n\gamma m_2 g(L)]^2}{n[\gamma m_2 g(L)]^2 + \gamma(1-\gamma)n[m_2 g(L)]^2 + d_2 n\gamma g^2(L) + d(L)n\gamma m_2} \tag{14}$$

Based on the definition of the noise factor (11) and using formulae (12)-(14) the noise factor for the three stages of amplification can be written as

$$F = \gamma^{-1}(1 + v_2 + v_3/m_2)$$

where $v_2 = d_2/m_2^2$ is the relative variance of the distribution at the output of the second stage, and $v_3 = d(L)/g^2(L)$ is the relative variance of the stage of amplification of a single electron.

5.2 The noise factor of an array of the channels

Let us define the noise factor at the output of the system of n parallel channels (Fig.1 [14]) where diameters of the channels are not necessarily the same. The single channel can be defined as one of n parallel amplification paths with the mean G_k and the variance D_k of the amplitude distribution at the output of the k-th channel. Let P_k be the probability of entering the channel with radius R_k. The part of the front surface of the channel plate closed to electrons is considered as a separate path with the mean G_0, variance D_0 and probability $P_0 = 1 - \gamma$.

Using (4) and (6) the mean G and variance D of the distribution at the output of the system of n channels can be obtained as:

$$G = \sum_{k=1}^{n} \gamma P_k G_k + (1-\gamma)G_0$$

$$D = \sum_{k=1}^{n} \gamma P_k D_k + (1-\gamma)D_0 + \sum_{k=1}^{n} \gamma P_k G_k^2 + (1-\gamma)G_0 - G^2$$

For $G_0 = 0$ and $D_0 = 0$ the noise factor F can be written as:

$$F = 1 + \frac{D}{G^2} = \frac{1}{\gamma}(1 + \frac{D_1}{G_1^2}) \tag{15}$$

where

$$G_1 = \sum_{k=1}^{n} P_k D_k$$

$$D_1 = \sum_{k=1}^{n} P_k D_k + \sum_{k=1}^{n} P_k G_k^2 - G_1^2$$

The distribution of the radii in the system of parallel channels is a continuous function, and the probability P_k should be changed to the probability density function $\Phi(R)$. Then, the expressions for G_1 and D_1 can be written as:

$$G_1 = \int_{R_{min}}^{R_{max}} \Phi(R) G(R) dR \tag{16}$$

$$D_1 = \int_{R_{min}}^{R_{max}} \Phi(R) D(R) dR + \int_{R_{min}}^{R_{max}} \Phi(R) G^2(R) dR - G_1^2 \tag{17}$$

6 Effect of Variations in Channel Diameters in the Channel Plate on the Noise Factor

Variations of the channel diameters as a result of technological distortions of a channel's geometry leads to the variations of the amplitude distributions at the outputs of different channels, and therefore it increases the noise factor.

The effect of the variations of the diameters in the channel plate on the noise factor can be evaluated using the equations (15) - (17). To find the probability density function $\Phi(R)$, let N be the total number of channels, n_k is the number of channels with the radius R_k, and $p_k = n_k/N$ is the probability of choosing the channel with radius R_k (probability distribution of the radii in the array of channels). If S_k is the area of the channels with radius R_k, and S is the total area of all channels, then the probability of entering the channel with radius R_k is defined by:

$$P_k = \frac{S_k}{S} \quad = \quad \frac{p_k R_k^2}{p_1 R_1^2 + p_2 R_2^2 + ...}$$

Therefore, the probability density distribution $\Phi(R)$ can be written as

$$\Phi(R) = \frac{R^2 \varphi(R)}{\int_0^\infty R^2 \varphi(R) dR} \tag{18}$$

where $\varphi(R)$ is the probability density distribution of the channels' radii in the array.

Assume that $\varphi(R)$ is defined by the probability density function for the normal distribution:

$$\varphi(R) = \frac{1}{\sigma_x\sqrt{2\pi}} exp[-\frac{(R-\bar{R})^2}{2\sigma_x^2}] \tag{19}$$

where σ_x is the variance, and $\bar{R}$ is the mean.

Substituting (19) into the equation (18) we obtain

$$\Phi(R) = AR^2 exp[-\frac{(R-\bar{R})^2}{2\sigma_x^2}]$$

where

$$A = [\sigma_x\sqrt{2\pi}\int_0^\infty R^2\varphi(R)dR]^{-1}$$

According to the normalization condition:

$$A\int_0^\infty R^2 exp[-\frac{(R-\bar{R})^2}{2\sigma_x^2}] = 1 \tag{20}$$

Introducing a new variable $t = \frac{R-\bar{R}}{\sigma_x}$ the expression (20) can be written as:

$$A(\sigma_x^3\int_{-\bar{R}/\sigma_x}^\infty t^2e^{-t^2/2}dt + 2\bar{R}\sigma_x^2\int_{-\bar{R}/\sigma_x}^\infty te^{-t^2/2}dt + \bar{R}^2\sigma_x\int_{-\bar{R}/\sigma_x}^\infty e^{-t^2/2}dt) = 1 \tag{21}$$

The integrals in (21) can be expressed via the integral:

$$V(x) = 2/\sqrt{2\pi}\int_0^x e^{-t^2/2}dt$$

Therefore, we obtain

$$\int_{-\bar{R}/\sigma_x}^\infty e^{-t^2/2}dt = \frac{\sqrt{2\pi}}{2}[1 + V(\frac{\bar{R}}{\sigma_x})]$$

$$\int_{-\bar{R}/\sigma_x}^\infty te^{-t^2/2}dt = e^{-\bar{R}^2/2\sigma_x^2}$$

$$\int_{-\bar{R}/\sigma_x}^\infty t^2e^{-t^2/2}dt = \frac{\sqrt{2\pi}}{2}[1 + V(\frac{\bar{R}}{\sigma_x})] - \frac{\bar{R}}{\sigma_x}e^{-\bar{R}^2/2\sigma_x^2}$$

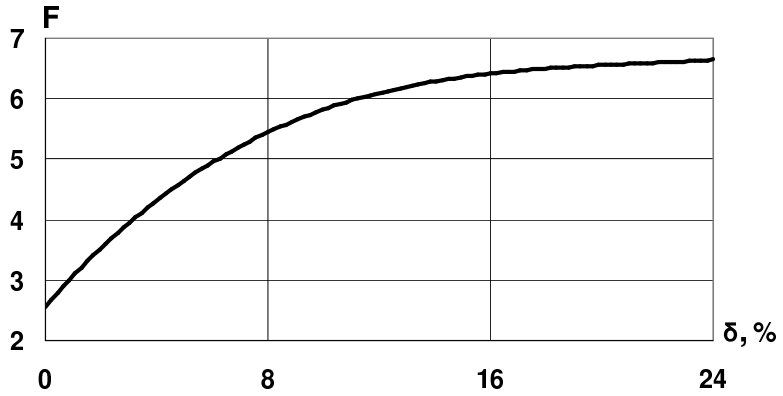

Figure 10: The noise factor as a function of the variations of the channel diameters

Finally, the expression for the probability density distribution $\Phi(R)$ can be written as:

$$\Phi(R) = \frac{2R^2 exp[-(R-\bar{R})^2/2\sigma_x^2]}{2\bar{R}\sigma_x^2 exp(-R^2/2\sigma_x^2) + \sigma_x\sqrt{2\pi}(\bar{R}^2+\sigma_x^2)[1+V(\bar{R}/\sigma_x)]}$$

For the normal distribution, the maximum and minimum radii of the channel in the array can be given as $R_{max} = \bar{R} + 3\sigma_x$ and $R_{min} = \bar{R} - 3\sigma_x$.

If δ is the variation of the channels' radii (as a percentage of the mean value $\bar{R}$) then $R_{max} = \bar{R} + \delta\bar{R}/100$ and $R_{min} = \bar{R} - \delta\bar{R}/100$.

Finally the probability density function $\Phi(R)$ is defined as

$$\Phi(R) = \frac{R^2 exp[-(R-\bar{R})^2/2\sigma_x^2]}{\bar{R}\sigma_x^2 exp(-R^2/2\sigma_x^2) + \sigma_x\sqrt{2\pi}(\bar{R}^2+\sigma_x^2)} \tag{22}$$

Fig.10 shows the noise factor $F(\delta)$ as a function of the variations of the channels' diameters δ for the mean $\bar{R} = 4\mu m$. It is seen that even for $\delta = 5\%$ of the mean value $\bar{R}$, the noise factor F increases by 75% compared with the noise factor for $\delta = 0$. [1]

Calculations of the noise factor as a function of the variations of the channels' diameters using only MC simulations would take about 3 days and nights of constant computer calculating. The use of the theorems reduces this time to 1 minute.

The results obtained here can be used to calculate the noise factor F for the given values of δ and $\bar{R}$, to calculate δ which provides the required

[1] It is clear that the variations of up to 25% about the mean channel radius is incompatible with the dimensions of real microchannel plates. However, the function $F(\delta)$ in Fig.10 is theoretical, and its behavior can be interesting from a mathematical point of view. The author would be happy to discuss the results with people who can provide similar experimental data.

value F, and also to optimize parameters of the channel plate in terms of the minimum F.

7 Conclusions

The method for calculation of the stochastic processes has been developed where the entire process is represented in the form of the sequence of several stages.

The theorems for the multistep sequential processes and for the parallel amplification paths have been proved. Expressions for the mean and variance of the amplitude distribution at the output of the system have been obtained.

The method has been used for investigating of the effect of variations in channel diameters on the noise characteristics of the microchannel plate.

Expressions for calculating the noise factor have been obtained. The noise factor, as a function of the variations of the channel diameters, has been calculated.

It has been shown that the amplitude distribution at the output of the channel is determined by the effective length of the channel. It enables MC simulations to be carried out only along this length for one electron, emitted at the beginning of the channel. The output of the entire amplification process is calculated by the use of the theorems and obtained characteristics. Any further investigations and optimizations can be conducted without MC simulations. The method provides highly accurate results and significantly reduces the cost of calculations. It enables the contribution of different amplification stages to the entire stochastic process to be easily investigated.

The theorems also can be applied to electronic devices where the channel multiplier is considered as one stage in the whole amplification process [12]. Moreover, the developed method is not limited by applications to only systems with the channel multipliers and can be used for many stochastic processes which require computer simulations.

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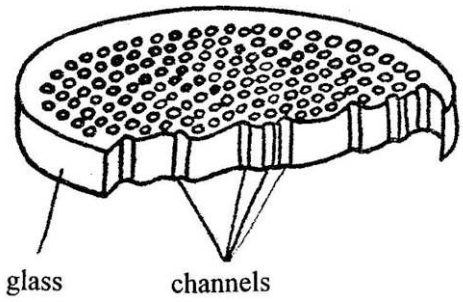
List of Figures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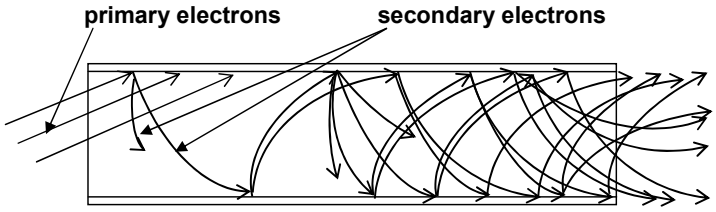
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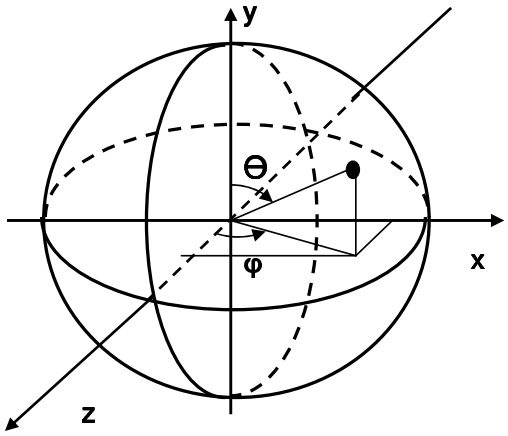
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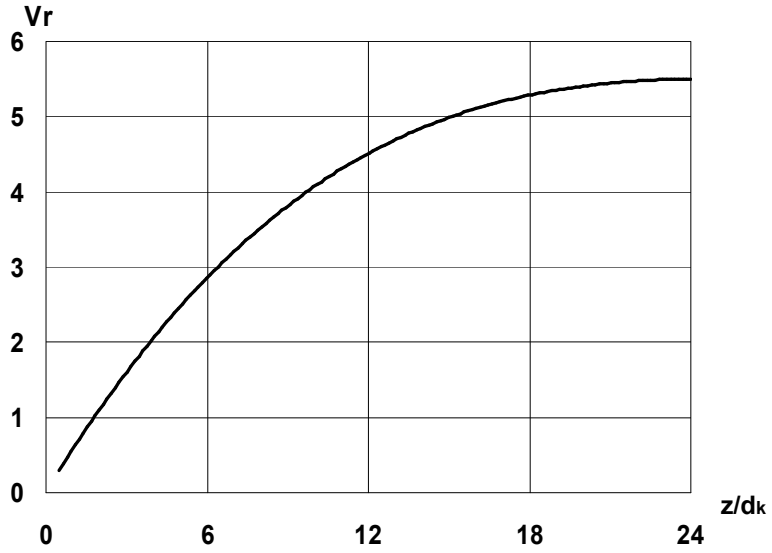
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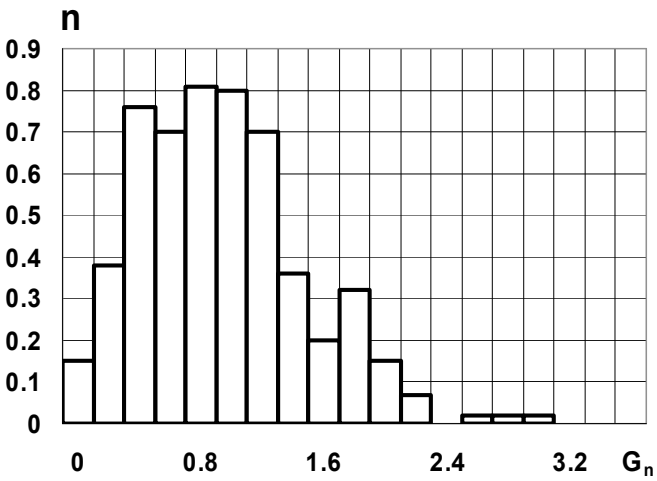
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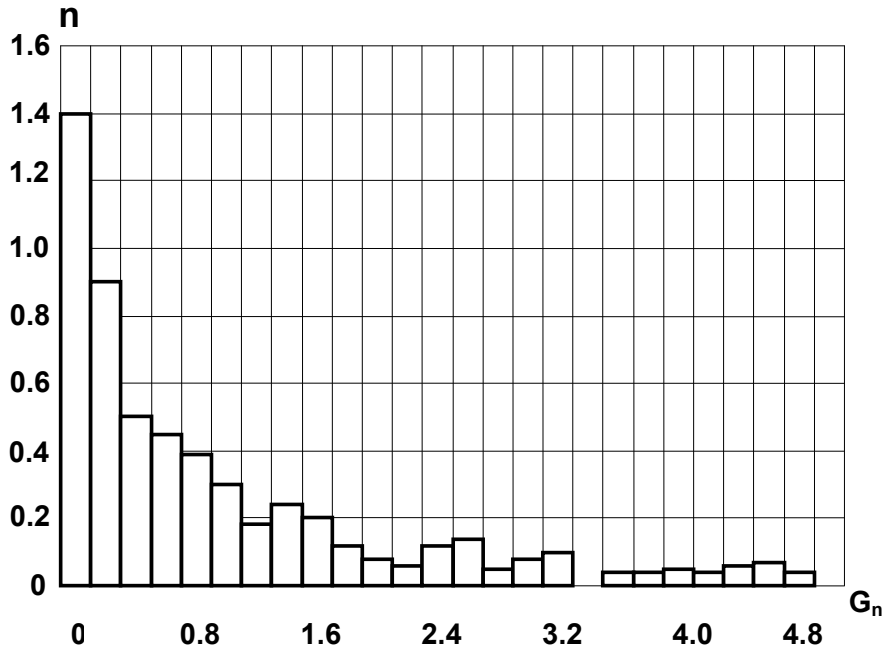
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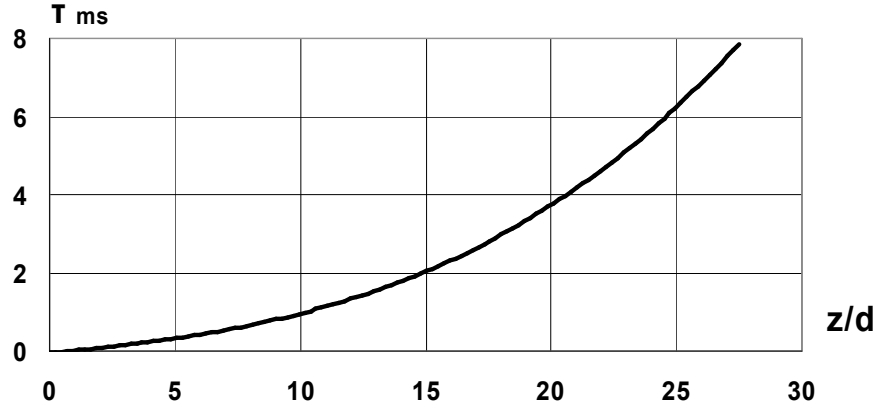
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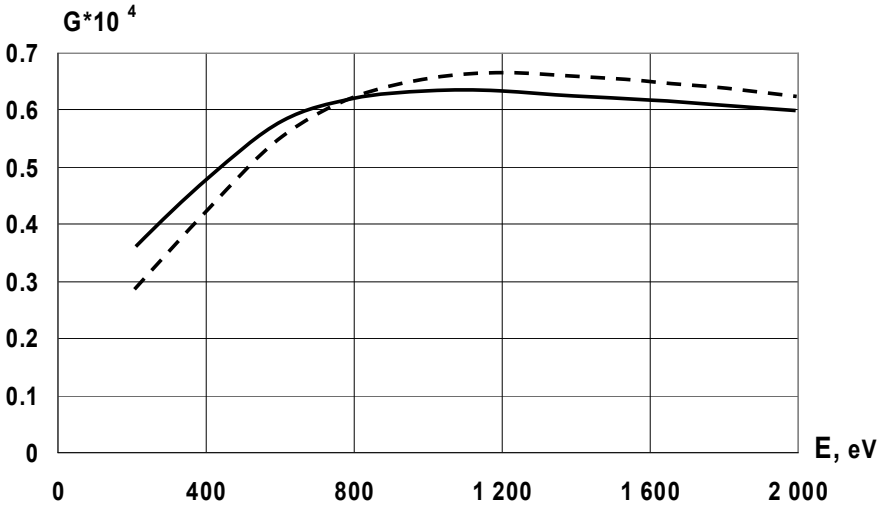
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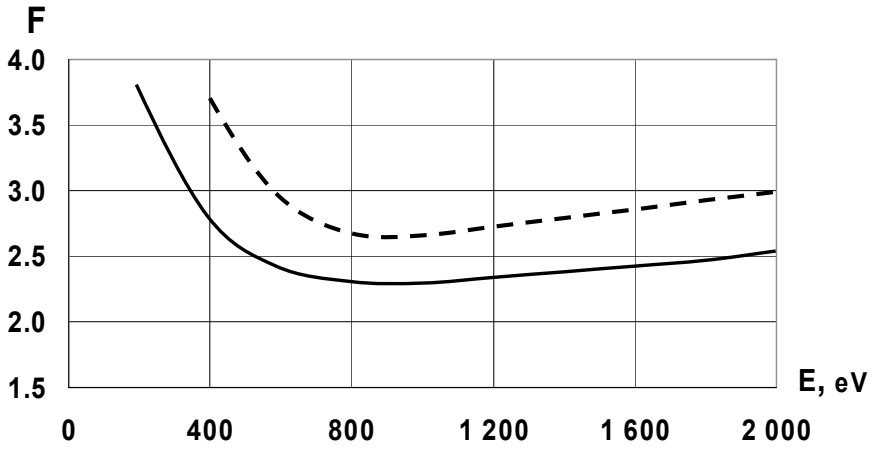
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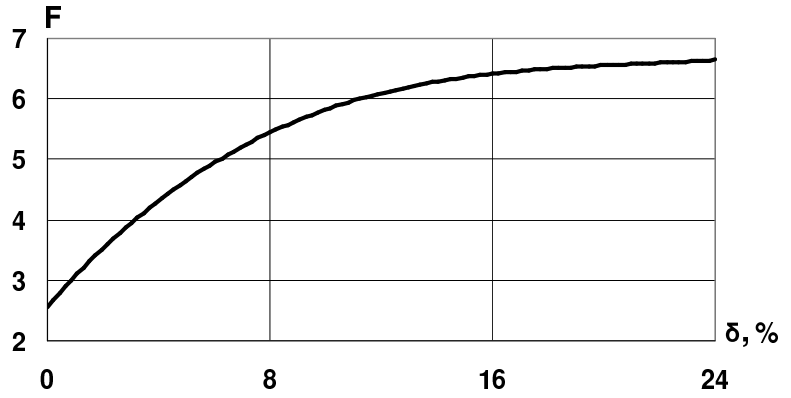
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